Math 5C Discussion Problems 3 Selected Solutions

Power Series

1. Find the radius and interval of convergence for the following power series.

   (a) $\sum 5^n x^n / n^n$ $\quad R = \infty$

   (b) $\sum x^n / \ln n$ $\quad R = 1$

   (c) $\sum (-1)^{n+3} x^{2n+1} / (n!)^2$ $\quad R = \infty$

   (d) $\sum x^n / (1 + n^2)$ $\quad R = 1$

   (e) $\sum x^n / (\ln n)^n$ $\quad R = \infty$

2. Remember the result: if $|y| < |x|$ and $\sum a_n x^n$ converges, then $\sum a_n y^n$ converges absolutely.

3. Assume that $\sum a_n 4^n$ converges conditionally. What can you say about the convergence of each of the following?

   (a) $\sum a_n 5^n$ $\quad$ diverges

   (b) $\sum a_n 2^n$ $\quad$ converges absolutely

   (c) $\sum a_n (-3)^n$ $\quad$ converges absolutely

   (d) $\sum a_n$ $\quad$ converges absolutely

4. Given that $f(x) = \sum a_n x^n$ converges in a neighborhood of zero, what is $f'''(0)$? $\quad 6a_3$

5. Which of the following are true for x near zero?

   (a) $x^2 = O(x^3)$ $\quad$ no

   (b) $x^3 = O(x^2)$ $\quad$ yes

   (c) $x^2 + x^4 = O(x^3)$ $\quad$ no

6. Note that

$$y^2 = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2 = x^2 + \frac{x \cdot x^2}{2!} + \frac{x^2 \cdot x}{2!} + O(x^4) = x^2 + x^3 + O(x^4).$$

7. The answer is

$$z = x - \frac{x^2}{2} + O(x^3).$$

Taylor Series

1. Done in lecture.

2. Use the series.

4. Evaluate the following limits.

   (a) $\lim_{x\to 0} \frac{\sin x^3}{x^3} = 1$

   (b) $\lim_{n\to\infty} n(e^{1/n} - 1) = 1$

   (c) $\lim_{x\to 0} \frac{\cos(\sin x) - \cos x}{\arctan(x^4)} = \frac{1}{6}$

   (d) $\lim_{x\to 0} \frac{e^{-x^2}\cos x - 1}{x^2} = -\frac{3}{2}$

   (e) $\lim_{x\to 0+} \frac{x^x - 1}{x \ln x} = 1$

   (f) $\lim_{x\to 0} \frac{e^{\sin x} - 1 - x}{x^2} = \frac{1}{2}$

   (g) $\lim_{n\to\infty} n^3 \left(\arctan\left(\frac{1}{n}\right) - \sin\left(\frac{1}{n}\right)\right) = -\frac{1}{6}$

   (h) Call the answer L.

$$L = \exp \lim_{n\to\infty} \ln \left[\left(1 + \frac{1}{n}\right)^{n^2} e^{-n}\right] = \exp \lim_{n\to\infty} \left[n^2 \ln\left(1 + \frac{1}{n}\right) - n\right]$$

   Use $\ln(1+x) = x - x^2/2 + O(x^3)$ to write

$$L = \exp \lim_{n\to\infty} \left[n^2\left(\frac{1}{n} - \frac{1}{2n^2} + O(n^{-3})\right) - n\right] = \exp \lim_{n\to\infty} \left[-\frac{1}{2} + O(n^{-1})\right] = e^{-1/2}$$

3. Either recognize known power series or manipulate known power series.

   (a) $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e$

   (b) $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots = \sin\pi = 0$

   (c) $1 + \frac{1}{2\cdot 3!} + \frac{1}{4\cdot 5!} + \frac{1}{8\cdot 7!} + \cdots = \sqrt{2}\sinh(1/\sqrt{2})$

   (d) Let $f(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots$ and note that we want $f(1/3)$. Now notice the derivative of the geometric series:

$$f(x) = \frac{d}{dx}\left(1 + x + x^2 + x^3 + \cdots\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2},$$

   so $f(1/3) = 9/4$.

   (e) $1 + \frac{2^2}{7} + \frac{3^2}{7^2} + \frac{4^2}{7^3} + \frac{5^2}{7^4} + \cdots = \frac{d}{dx}\sum_{n=0}^{\infty} nx^n\Big|_{x=1/7} = \frac{d}{dx}\frac{x}{(1-x)^2}\Big|_{x=1/7} = \frac{49}{27}$

   (f) $1 + \frac{1}{2\cdot 7} + \frac{1}{3\cdot 7^2} + \frac{1}{4\cdot 7^3} + \frac{1}{5\cdot 7^4} + \cdots = 7\ln(7/6)$

Fourier Series

1. Think orthogonality!

2. For each of the following 2π-periodic functions, compute the Fourier series.

   (a) Since f is odd, all $a_n = 0$. Then $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2(1-(-1)^n)}{\pi n}$. Thus

   $$f(x) \sim \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{\pi n} \sin nx$$

   (b) Since f is odd, all $a_n = 0$. Then $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2(-1)^{n+1}}{n}$. Thus

   $$f(x) \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

   (c) Just compute the integrals. No tricks.

   (d) Again, just compute.

3. Let $h(x) = f(x)g(x)$. If f, g are even then $h(-x) = f(-x)g(-x) = f(x)g(x) = h(x)$. If f, g are odd then $h(-x) = f(-x)g(-x) = (-1)^2 f(x)g(x) = h(x)$. If f is even and g is odd, then $h(-x) = f(-x)g(-x) = -f(x)g(x) = -h(x)$.

4. Substitute $u = -x$ into the integrals for a_n, b_n.

7. *Solution.* Replacing x with $|x|$ makes the identity true for $-\pi < x < \pi$. Parseval gives

   $$\frac{1}{\pi}\int_{-\pi}^{\pi} \ln^2\left(2\sin\frac{|x|}{2}\right) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

   The integrand is an even function, so

   $$\int_0^{\pi} \ln^2\left(2\sin\frac{x}{2}\right)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \ln^2\left(2\sin\frac{|x|}{2}\right)\,dx = \frac{\pi^3}{12}. \qquad \square$$

Series: Miscellany

1. *Solution.* For small t, $\ln(1+t) = t - t^2/2 + O(t^3)$. Thus

$$\ln\left(1+\frac{1}{n}\right)^n = n\ln\left(1+\frac{1}{n}\right) = 1 - \frac{1}{2n} + O(n^{-2})$$

Undoing the logarithm,

$$\left(1+\frac{1}{n}\right)^n = \exp\left(1 - \frac{1}{2n} + O(n^{-2})\right) = e\cdot\exp\left(-\frac{1}{2n} + O(n^{-2})\right)$$

For small t, $\exp(t) = 1 + t + O(t^2)$, so

$$\left(1+\frac{1}{n}\right)^n = e\cdot\left[1 + \left(-\frac{1}{2n} + O(n^{-2})\right) + O(n^{-2})\right] = e - \frac{e}{2n} + O(n^{-2})$$

The limit is

$$\lim_{n\to\infty} n\left(\left(1+\frac{1}{n}\right)^n - e\right) = \lim_{n\to\infty} n\left(-\frac{e}{2n} + O(n^{-2})\right) = \lim_{n\to\infty} -\frac{e}{2} + O(n^{-1}) = -\frac{e}{2}$$

In other words,

$$\left(1+\frac{1}{n}\right)^n = e - \frac{e}{2n} + O(n^{-2})$$

for large n. □

2. *Hint.* Use the fact that $e = 1/0! + 1/1! + 1/2! + \cdots$ □

3. Use what you know about approximating functions to determine convergence of these series.

   (a) $\sum\left(1-\cos\left(\frac{1}{n}\right)\right) \approx \sum \frac{1}{2n^2}$. Use limit comparison to make rigorous.

   (b) $\sum\left(1-n\sin\left(\frac{1}{n}\right)\right) \approx \sum \frac{1}{6n^2}$

Harmonic Functions

1. Which of these functions are harmonic on $\mathbb{R}^2$?

   (a) $f(x, y) = e^x \sin y$ (yes)

   (b) $f(x, y) = x^2 - y^2$ (yes)

   (c) $f(x, y) = x^2 + y^2$ (no)

   (d) $f(x, y) = \ln(x^2 + y^2)$ (yes, unless you want to argue that it isn't defined at the origin)

2. If $f = g = x$, then $\Delta f = \Delta g = 0$, but $\Delta(fg) = 2$.

4. Assume B is a ball and f and g are harmonic. If $f = g$ on ∂B, then $f - g$ is harmonic and identically 0 on ∂B. Since $f - g$ attains its max and min on the boundary, $f - g \leq 0$ in B and $f - g \geq 0$ in B. Thus $f - g = 0$ everywhere in B.

7. Suppose f is harmonic and $f = 0$ on ∂B, where B is a ball. The zero boundary condition implies the surface integral below is zero:

$$0 = \iint_{\partial B} f \nabla f \cdot d\mathbf{A} = \iiint_B \|\nabla f\|^2 \, dV.$$

   Since $\|\nabla f\|^2 \geq 0$, the integral could only be zero if $\nabla f = 0$ everywhere. Thus f is constant.

8. Again, $f - g$ is harmonic and zero on the boundary.

9. Find constants A and B so that $A \ln(x^2 + y^2) + B$ matches the boundary data.

10. f must be constant!

12. Since f is harmonic, use the mean-value property.

$$\iint_{\partial B} f \, d\sigma = \text{area}(\partial B) f(10, 10, 10) = 40\pi \ln 200.$$